

# AE solutions to interval max-plus matrix equations

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## Abstract:

Max-plus algebra is an algebraic structure which classical addition and multiplication are replaced by the operations  $\oplus$  and  $\otimes$ . This study extends the concepts of AE solutions for interval linear system to AE solutions for interval matrix equations (IME) within the framework of max-plus algebra. The research provides a comprehensive characterization of AE solutions to interval matrix equations, covering both linear inequalities and linear equations. It introduces general framework that includes specific solution types- weak, strong, tolerance and control solutions-as special cases, encompassing many known solution concepts for interval max-plus matrix equations. Additionally, the study presents specific examples to demonstrate an efficient method for verifying AE solutions to interval max-plus matrix equations (MPME) and highlights results for the various solution types, including weak, strong, tolerance and control solutions.

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## 1. Introduction

Max-plus algebra ( $\mathbb{R}_{\max}, \oplus, \otimes$ ) has appeared under the name extremal algebra for numerous years [1, 2]. In max-plus algebra, there are several types of solvability of interval linear systems, including both interval linear equations and interval linear inequalities which have been reported in several literatures. For example weak and strong solvability were reported in Butkovic, *et al.* [3] and Lihua, *et al.* [4], tolerance solvability was reported in Butkovic [5, 6, 7] while control solvability was reported in Butkovic [8].

Max-plus algebra is one of many idempotent semi-rings which have been considered in various fields of mathematics. It appeared in P. Butkovic paper on nerve sets and automata [9, 10]. It has found applications in many areas such as combinatorial, optimization, mathematical physics and algebraic geometry [10]. It is also used in control theory, machine scheduling, discrete event processes, manufacturing systems, telecommunication networks, parallel processing systems and traffic control [11, 12, 13]. Many equations that are used to describe the behavior of these applications are nonlinear in conventional algebra but become linear in max-plus algebra. This is a primary reason for its utility in various areas [14]. Many of the theorems and techniques used in classical linear algebra have analogues in the max-plus semi-ring. Researchers such as Allamigeon, *et al.* [13], Butkovic [14] and Butkovic, *et al.* [15] have studied concepts such as solving systems of linear equations, the eigenvalue problem, and linear independence in the max-plus sense and reported. AE-solution sets are generalized

solution sets to interval equations (inequalities, etc.) for which the separating predicate has AE-form.

In this paper, the concept of AE solutions of interval linear system was extended to AE solutions of IME over max-plus algebra. Necessary and sufficient conditions for checking the AE solvability of matrix equation of interval linear inequalities in max-plus algebra was formulated. Then, fully characterized AE solutions of IME over max-plus algebra were result types such as; weak, strong, tolerance and control solutions was discussed.

## 2. Definitions, Notations and algebraic properties

Some key definitions and algebraic properties of max-plus algebra were presented as reported in several literatures among which include Butkovic, *et al.* [16], Akian *et al.* [17] and Aminu and Butkovic [18]. They reported that in max-plus algebra for  $a, b \in \mathbb{R}_{\max} \setminus \{-\infty\} \cup \{\mathbb{R}\} = \overline{\mathbb{R}}$ , two operations were defined  $\oplus$  and  $\otimes$  by

$$a \oplus b = \max\{a, b\}, a \otimes b = a + b \quad (1)$$

In max-plus algebra, also a triple  $(\mathbb{R}_{\max}, \oplus, \otimes)$ , where  $\mathbb{R}_{\max} = \mathbb{R} \cup \{\varepsilon\}$ ,  $\mathbb{R}$  is the set of real numbers, where  $\varepsilon = -\infty$ . Let  $\xi$  be the matrix consisting entirely of  $\varepsilon$ . The set of all  $m \times n$  matrices over  $\overline{\mathbb{R}}$  is denoted by  $\overline{\mathbb{R}}(m, n)$  and the set of all column  $n$  - vector over  $\overline{\mathbb{R}}$  by  $\overline{\mathbb{R}}(n)$ .

Operations  $\oplus, \otimes$  are extended to matrices and vectors in the same way as in the classical algebra. In max-plus algebra  $e = 0$  is the multiplicative identity:

$$a \otimes e = e \otimes a = a + 0 = a \text{ for all } a \in \mathbb{R}_{\max} \quad (2)$$

The additive identity is  $\epsilon = -\infty$

$$a \oplus \epsilon = \epsilon \oplus a = \max(a, -\infty) = a \text{ for } a \in \mathbb{R}_{\max} \quad (3)$$

Clearly  $\oplus$  and  $\otimes$  are commutative, and obey many other properties similar to  $+$  and  $\times$  in conventional algebra. For instance, it can be seen that  $\otimes$  distributes over  $\oplus$  as follows: for  $a, b, c \in \mathbb{R}_{\max}$  there will be:

$$\begin{aligned} a \otimes (b \oplus c) &= a + \max(b, c) = \max(a + b, a + c) \\ &= (a \otimes b) \oplus (a \otimes c) \end{aligned} \quad (4)$$

This and other basic algebraic properties are listed in the following lemmas. The proofs are elementary, very short and precise.

## 2.1 Lemma 1:

As reported in Butkovič and Jones [6], for all  $x, y, z \in \mathbb{R}_{\max}$ , then

- Associativity  $x \oplus (y \oplus z) = (x \oplus y) \oplus z$  and  $x \otimes (y \otimes z) = (x \otimes y) \otimes z$
- Commutability  $x \oplus y = y \oplus x$  and  $x \otimes y = y \otimes x$
- Distributivity  $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$
- Additive Identity  $\epsilon = -\infty, x \oplus \epsilon = \epsilon \oplus x = x$
- Multiplicative Identity  $e = 0: x \otimes e = e \otimes x = x$
- Multiplicative Inverse; if  $x \neq e$ , then  $\exists$  a distinct  $y$  with  $x \otimes y = e$
- Unit Element  $x \otimes e = e \otimes x = x$
- Absorbing Element  $x \otimes \epsilon = \epsilon \otimes x = \epsilon$ .

Proofs:

### a. For Associativity under $(\mathbb{R}_{\max}, \oplus)$

$$\begin{aligned} x \oplus (y \oplus z) &= x \oplus \max(y, z) \\ &= \max(x, \max(y, z)) = \max(x, y) \oplus z = (x \oplus y) \oplus z \end{aligned}$$

Under  $(\mathbb{R}_{\max}, \otimes)$

$$\begin{aligned} x \otimes (y \otimes z) &= x + (y \otimes z) = x + (y + z) \\ &= (x + y) + z = (x \otimes y) \otimes z \end{aligned} \quad (5)$$

### b. Commutability

$$\begin{aligned} x \oplus y &= \max(x, y) = \max(y, x) = y \oplus x \\ x \otimes y &= x + y = y + x = y \otimes x \end{aligned} \quad (6)$$

**Note:** But when dealing with matrices, let say  $X$  and  $Y$ , so they are only commutative when  $X=Y$  or one of them is a unit matrix, as we can see example in the next section.

### c. Distributivity

$$\begin{aligned} x \otimes (y \oplus z) &= x + \max(y, z) \\ &= \max(x + y, x + z) = (x \otimes y) \oplus (x \otimes z) \end{aligned} \quad (7)$$

### d. Additive Identity

Since  $\epsilon = -\infty$ , then

$$\begin{aligned} x \oplus \epsilon &= \epsilon \oplus x = \max(x, -\infty) \\ &= \max(-\infty, x) = x \end{aligned} \quad (8)$$

### e. Multiplicative Identity

$$e = 0: x \otimes e = e \otimes x = 0 + x = x$$

## f. Multiplicative Inverse

if  $x \neq e$ , then  $\exists$  a distinct  $y$  with  $x \otimes y = e$  (9)

## g. Unit Element

$$x \otimes e = e \otimes x = x + 0 = x \quad (10)$$

## h. Absorbing Element

$$\begin{aligned} x \otimes \epsilon &= \epsilon \otimes x = x \otimes (-\infty) = x + (-\infty) \\ &= (-\infty) = \epsilon \end{aligned} \quad (11)$$

Hence the additive Identity  $\epsilon$  is absorbing under multiplication.

**Definition 1:** Considering the ordering " $\leq$ " on the sets  $\overline{\mathbb{R}}(m, n)$  and  $\overline{\mathbb{R}}(n)$  defined as follows:

- for  $A, B \in \overline{\mathbb{R}}(m, n)$ :  $A \leq B$  if  $a_{ij} \leq b_{ij}$  for each  $i \in M$  and each  $j \in N$ ,
- for  $x, y \in \overline{\mathbb{R}}(n)$ :  $x \leq y$  if  $x_j \leq y_j$  for each  $j \in N$ .

## 2.2 Lemma 2:

As reported in Butkovič and Jones [6], if  $A, B$ , and  $C$  are matrices over  $\overline{\mathbb{R}}$  of compatibles size and  $c \in \overline{\mathbb{R}}$  then,

$$A \geq B \Rightarrow A \oplus C \geq B \oplus C,$$

$$A \geq B \Rightarrow A \otimes C \geq B \otimes C$$

$$A \geq B \Rightarrow C \otimes A \geq C \otimes B$$

$$A \geq B \Rightarrow c \otimes A \geq c \otimes B$$

**Definition 2:** Using the monotonicity of  $\otimes$ , which means that for each  $A, C \in \overline{\mathbb{R}}(m, n)$  and  $B, D \in \overline{\mathbb{R}}(n, s)$  the implication. If

$A \leq C$  and  $B \leq D$  then  $A \otimes B \leq C \otimes D$  holds.

In the following, shall use the notion of a residuated function see [2, 6].

If  $A \in \overline{\mathbb{R}}(m, n)$ ,  $B \in \overline{\mathbb{R}}(m, r)$ , define the sum as  $A \oplus B \in \overline{\mathbb{R}}(m, n)$  With entries  $(A \oplus B)_{ij}$  defined for  $i = 1, 2, 3, \dots, m, j = 1, 2, 3, \dots, n$ . as follows:

$$(A \oplus B)_{ij} = \max\{a_{ij}, b_{ij}\}.$$

## 3. Numerical Examples of max-plus on Matrix Operations

$$\text{Let } A = \begin{bmatrix} 4 & 0 \\ -2 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ -\infty & 8 \end{bmatrix} \text{ and } x = 3, \quad (12)$$

Where  $A, B \in \mathbb{R}_{\max}$

### 3.1 Matrix addition $(A \oplus B)$

$$A \oplus B = \begin{bmatrix} 4 & 0 \\ -2 & 5 \end{bmatrix} \oplus \begin{bmatrix} 3 & 1 \\ -\infty & 8 \end{bmatrix}$$

$$A \oplus B = \begin{bmatrix} 4 \oplus 3 & 0 \oplus 1 \\ -2 \oplus -\infty & 5 \oplus 8 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -2 & 8 \end{bmatrix}$$

$$A \oplus B = \begin{bmatrix} 4 & 1 \\ -2 & 8 \end{bmatrix}.$$

### 3.2 Scalar Multiplication ( $x \otimes A$ )

$$x \otimes A = 3 \otimes \begin{bmatrix} 4 & 0 \\ -2 & 5 \end{bmatrix}$$

$$x \otimes A = \begin{bmatrix} 3 \otimes 4 & 3 \otimes 0 \\ 3 \otimes -2 & 3 \otimes 5 \end{bmatrix} = \begin{bmatrix} 3 + 4 & 3 + 0 \\ 3 + (-2) & 3 + 5 \end{bmatrix}$$

$$x \otimes A = \begin{bmatrix} 7 & 3 \\ 1 & 8 \end{bmatrix}$$

### 3.3 Matrix multiplication ( $A \otimes B$ )

$$A \otimes B = \begin{bmatrix} 4 & 0 \\ -2 & 5 \end{bmatrix} \otimes \begin{bmatrix} 3 & 1 \\ -\infty & 8 \end{bmatrix}$$

$$= \begin{bmatrix} (4 \otimes 3) \oplus (0 \otimes -\infty) & (4 \otimes 1) \oplus (0 \otimes 8) \\ (-2 \otimes 3) \oplus (5 \otimes -\infty) & (-2 \otimes 1) \oplus (5 \otimes 8) \end{bmatrix}$$

$$= \begin{bmatrix} 7 \oplus -\infty & 5 \oplus 8 \\ 1 \oplus -\infty & -1 \oplus 13 \end{bmatrix}$$

$$B \otimes A = \begin{bmatrix} 7 & 8 \\ 1 & 13 \end{bmatrix}$$

### 3.4 Matrix multiplication ( $B \otimes A$ )

$$B \otimes A = \begin{bmatrix} 3 & 1 \\ -\infty & 8 \end{bmatrix} \otimes \begin{bmatrix} 4 & 0 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} (3 \otimes 4) \oplus (1 \otimes -2) & (3 \otimes 0) \oplus (1 \otimes 5) \\ (-\infty \otimes 4) \oplus (8 \otimes -2) & (-\infty \otimes 0) \oplus (8 \otimes 5) \end{bmatrix}$$

$$= \begin{bmatrix} 7 \oplus -1 & 3 \oplus 6 \\ -\infty \oplus 6 & -\infty \oplus 13 \end{bmatrix}$$

$$B \otimes A = \begin{bmatrix} 7 & 6 \\ 6 & 13 \end{bmatrix}$$

It is notice that multiplication of matrices in  $(\mathbb{R}_{\max}, \otimes)$  is associative, that is  $A \otimes (B \otimes C) = (A \otimes B) \otimes C$  but not commutative, thus,  $A \otimes B \neq B \otimes A$ . It is only commutative when  $A = B$  or when one of the matrices is a unit matrix. This is when the entries of A, B and C are from  $\mathbb{R}_{\max}$ .

**Definition 3: Matrix equations:** Let  $A \in \mathbb{R}(m, n)$ ,  $B \in \mathbb{R}(m, r)$ ,  $C \in \mathbb{R}(s, r)$  and an unknown matrix  $X \in \mathbb{R}(n, s)$ . We get the matrix equation of the form

$$A \otimes X = B \quad (13)$$

It is easy to see that equation 13 is equivalent to the following  $k$  equations:  $A \otimes X_k = B_k$

For each  $k \in R$ , where  $X_k(B_k)$  is a  $k$ -th column of  $X(B)$ .

Consequently the principal solution of equation 14 is the matrix  $X^*(A, B)$  with elements

$$x_{jk}^*(A, B) = x_j^*(A, B_k) \quad (14)$$

The topic of this paper is a solution of two-sided matrix equations of the form

$$A \otimes X \otimes C = B \quad (15)$$

Where  $A \in \mathbb{R}(m, n)$ ,  $B \in \mathbb{R}(m, r)$  and  $C \in \mathbb{R}(s, r)$  are given matrices and we are looking for  $X \in \mathbb{R}(n, s)$

The following Lemmas will be used in the proofs of our main results.

### 3.5 Lemma 3:

Let  $A = (a_{ij}) \in \mathbb{R}(m, n)$ ,  $B = (b_{ij}) \in \mathbb{R}(m, n)$ .

If  $A \leq B$ , then  $A \otimes X \leq B \otimes X$

**Proof:** Note that  $a_{ij} \leq b_{ij}$ , and for  $i = 1, \dots, m$ , we have

$$[A \otimes X]_i = \max_{1 \leq j \leq n} \{a_{ij} \otimes x_j\} = \max_{1 \leq j \leq n} \{a_{ij} + x_j\} \quad (17)$$

And

$$[B \otimes X]_i = \max_{1 \leq j \leq n} \{b_{ij} \otimes x_j\} = \max_{1 \leq j \leq n} \{b_{ij} + x_j\} = \max_{1 \leq j \leq n} \{b_{ij} + x_j\} \quad (18)$$

Thus,

$$[A \otimes X]_i \leq [B \otimes X]_i \text{ for } i = 1, \dots, m, \text{ i.e. } A \otimes X \leq B \otimes X.$$

### 3.6 Lemma 4:

Let  $A = (a_{ij}) \in \mathbb{R}(m, n)$ ,  $B = (b_{ij}) \in \mathbb{R}(m, n)$ ,

$C = (c_{ij}) \in \mathbb{R}(m, n)$ . If

$A \leq C$ , then  $A \oplus B \leq B \oplus C$ .

**Proof.** Note that  $a_{ij} \leq c_{ij}$ , we have

$$(A \oplus B)_{ij} = \max\{a_{ij}, b_{ij}\} \leq \max\{c_{ij}, b_{ij}\} = (B \oplus C)_{ij}$$

for  $i = 1, \dots, m, j = 1, \dots, n$  i.e.,  $A \oplus B \leq B \oplus C$ .

**Definition 4: Interval matrix equations:** Similarly, the studies of Butkovic [7, 9], Allamigeon, et al. [13], Butkovic and MacCaig [14], Butkovic, et al. [15] and Akian, et al. [17], defined interval matrices  $A, B, C$  as follows:

$$A = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}(m, n) | \underline{A} \leq A \leq \overline{A}\}$$

$$B = [\underline{B}, \overline{B}] = \{B \in \mathbb{R}(m, r) | \underline{B} \leq B \leq \overline{B}\}$$

$$C = [\underline{C}, \overline{C}] = \{C \in \mathbb{R}(s, r) | \underline{C} \leq C \leq \overline{C}\}.$$

Donated by

$$A \otimes X \otimes C = B \quad (19)$$

The set of all matrix equation of the form of (18):  $A \in A, B \in B, C \in C$ . Equation (19) is called the interval max-plus matrix equation

Where  $\underline{A}, \overline{A} \in \mathbb{R}(m, n)$ ,  $\underline{A} \leq \overline{A}$  and " $\leq$ " is understood componentwise. Similarly,

$$\underline{B}, \overline{B} \in \mathbb{R}(m, r) \underline{B} \leq \overline{B} \text{ and } " \leq ", \underline{C}, \overline{C} \in \mathbb{R}(s, r) \underline{C} \leq \overline{C} \text{ and } " \leq "$$

is component wise.

Given  $A \in \mathbb{R}(m, n)$ ,  $B \in \mathbb{R}(m, r)$ ,  $C \in \mathbb{R}(s, r)$  and an unknown matrix  $X$ , the corresponding interval linear system of inequalities

$$A \otimes X \otimes C \leq B \text{ Is the family of } A \otimes X \otimes C \leq B$$

Where  $A \in \mathbf{A}, B \in \mathbf{B}, C \in \mathbf{C}$  and  $X$  is unknown matrix.

In more general case, Let  $A \in \overline{\mathbb{R}}(m_1, n)$ ,  $B \in \overline{\mathbb{R}}(m_1, r)$ ,  $C \in \overline{\mathbb{R}}(s_1, r)$ , and  $D \in \overline{\mathbb{R}}(m_2, r)$ ,  $E \in \overline{\mathbb{R}}(m_2, r)$ ,  $F \in \overline{\mathbb{R}}(s_2, r)$ , the corresponding interval system of inequalities of the following form:

$$A \otimes X \otimes C \leq B \quad (20)$$

$$D \otimes X \otimes F \geq E \quad (21)$$

Represent the set of all linear systems of matrix max-plus inequalities of the form:

$$A \otimes X \otimes C \leq B \quad (22)$$

$$D \otimes X \otimes F \geq E \quad (23)$$

Where  $A \in \mathbf{A}, B \in \mathbf{B}, C \in \mathbf{C}$  and  $D \in \mathbf{D}, E \in \mathbf{E}, F \in \mathbf{F}$ .

Remark 1: Clearly, a real linear system  $(D \otimes F)X \geq E$  can be transformed to an equivalent system  $(-D \otimes -F)X \leq -E$ . However, linear max-plus system

$D \otimes X \otimes F \geq E$  and  $(-D \otimes -F)X \leq -E$  are not identical in general.

**Definition 5:** A matrix  $X$  is an AE solution to  $A \otimes X \otimes C = B$  if and only if

$$\forall A^\forall \in \mathbf{A}^\forall, \forall C^\forall \in \mathbf{C}^\forall, \forall B^\forall \in \mathbf{B}^\forall, \exists A^\exists \in \mathbf{A}^\exists, \exists C^\exists \in \mathbf{C}^\exists, \exists B^\exists \in \mathbf{B}^\exists$$

Such that

$$[(A^\forall \oplus A^\exists) \otimes (C^\forall \oplus C^\exists)] \otimes X = B^\forall \oplus B^\exists.$$

**Example 1:** Consider the interval matrices

$$A = \begin{pmatrix} [2,6] & [1,3] \\ [1,4] & [3,4] \end{pmatrix},$$

$$C = \begin{pmatrix} [2,5] & [0,4] \\ [3,7] & [7,9] \end{pmatrix},$$

$$B = \begin{pmatrix} [3,15] & [6,17] \\ [2,13] & [5,15] \end{pmatrix}$$

If  $X = \begin{pmatrix} [0,1] & [2,3] \\ [1,5] & [4,7] \end{pmatrix}$  then,  $X$  is an AE solution to

$$A \otimes X \otimes C = B$$

By **Definition 5:** it is need to show that

$$[(A^\forall \oplus A^\exists) \otimes (C^\forall \oplus C^\exists)] \otimes X = B^\forall \oplus B^\exists$$

From **Definition 5,**

$$A^\forall = \begin{pmatrix} [\varepsilon, \varepsilon] & [1,3] \\ [1,4] & [\varepsilon, \varepsilon] \end{pmatrix}, C^\forall = \begin{pmatrix} [\varepsilon, \varepsilon] & [0,4] \\ [3,7] & [\varepsilon, \varepsilon] \end{pmatrix},$$

$$A^\exists = \begin{pmatrix} [2,6] & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & [3,4] \end{pmatrix}, C^\exists = \begin{pmatrix} [2,5] & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & [7,9] \end{pmatrix}$$

And

$$B^\forall = \begin{pmatrix} [3,15] & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & [5,15] \end{pmatrix}, B^\exists = \begin{pmatrix} [\varepsilon, \varepsilon] & [6,17] \\ [2,13] & [\varepsilon, \varepsilon] \end{pmatrix},$$

Then the left hand side obtained as

$$\begin{aligned} & \left( \begin{pmatrix} [\varepsilon, \varepsilon] & [1,3] \\ [1,4] & [\varepsilon, \varepsilon] \end{pmatrix} \oplus \begin{pmatrix} [2,6] & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & [3,4] \end{pmatrix} \right) \otimes \\ & \left( \begin{pmatrix} [\varepsilon, \varepsilon] & [0,4] \\ [3,7] & [\varepsilon, \varepsilon] \end{pmatrix} \oplus \begin{pmatrix} [2,5] & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & [7,9] \end{pmatrix} \right) \otimes \\ & \begin{pmatrix} [0,1] & [2,3] \\ [1,5] & [4,7] \end{pmatrix} \\ & = \left( \begin{pmatrix} [2,6] & [1,3] \\ [1,4] & [3,4] \end{pmatrix} \otimes \begin{pmatrix} [2,5] & [0,4] \\ [3,7] & [7,9] \end{pmatrix} \right) \otimes \begin{pmatrix} [0,1] & [2,3] \\ [1,5] & [4,7] \end{pmatrix} \\ & = \begin{pmatrix} [2,6] \otimes [2,5] \oplus [1,3] \otimes [3,7] & [2,6] \otimes [0,4] \oplus [1,3] \otimes [7,9] \\ [1,4] \otimes [2,5] \oplus [3,4] \otimes [3,7] & [1,4] \otimes [0,4] \oplus [3,4] \otimes [7,9] \end{pmatrix} \\ & \otimes \begin{pmatrix} [0,1] & [2,3] \\ [1,5] & [4,7] \end{pmatrix} \\ & = \begin{pmatrix} [4,11] \oplus [4,10] & [2,10] \oplus [8,12] \\ [3,9] \oplus [6,11] & [1,8] \oplus [10,13] \end{pmatrix} \otimes \begin{pmatrix} [0,1] & [2,3] \\ [1,5] & [4,7] \end{pmatrix} \\ & = \begin{pmatrix} [4,11] & [2,10] \\ [3,9] & [1,8] \end{pmatrix} \otimes \begin{pmatrix} [0,1] & [2,3] \\ [1,5] & [4,7] \end{pmatrix} \\ & = \begin{pmatrix} [4,11] \otimes [0,1] \oplus [2,10] \otimes [1,5] & [4,11] \otimes [2,3] \oplus [2,10] \otimes [4,7] \\ [3,9] \otimes [0,1] \oplus [1,8] \otimes [1,5] & [3,9] \otimes [2,3] \oplus [1,8] \otimes [4,7] \end{pmatrix} \\ & = \begin{pmatrix} [4,12] \oplus [3,15] & [6,14] \oplus [6,17] \\ [3,10] \oplus [2,13] & [5,12] \oplus [5,15] \end{pmatrix} = \begin{pmatrix} [3,15] & [6,17] \\ [2,13] & [5,15] \end{pmatrix}. \end{aligned}$$

Similarly, the right hand side will be obtained as

$$\begin{aligned} B^\forall \oplus B^\exists & = \left( \begin{pmatrix} [3,15] & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & [5,15] \end{pmatrix} \oplus \begin{pmatrix} [\varepsilon, \varepsilon] & [6,17] \\ [2,13] & [\varepsilon, \varepsilon] \end{pmatrix} \right) \\ & = \begin{pmatrix} [3,15] & [6,17] \\ [2,13] & [5,15] \end{pmatrix} \end{aligned}$$

## 4. AE solutions to IME

### 4.1 IME over max-plus algebra

Each interval in interval linear system is associated with either the universal quantifier  $\forall$ , which is called A-uncertainty, or the existential quantifier  $\exists$ , which is called E-uncertainty. Therefore, let  $A \in \overline{\mathbb{R}}(m, n)$ ,  $B \in \overline{\mathbb{R}}(m, r)$  and  $C \in \overline{\mathbb{R}}(s, r)$  be given matrices and we are looking for  $X \in \overline{\mathbb{R}}(n, s)$ . Components in the matrix  $A^\forall$  on the positions associated with existential quantifier are intervals  $[0, 0]$ , and components in the matrix  $A^\exists$ , on the positions associated with universal quantifier are intervals  $[0, 0]$ . Thus, we can split the interval matrix as  $A = A^\forall + A^\exists$ , where  $A^\forall$  are the interval matrix comprising universally quantified coefficients, and  $A^\exists$  concerns existentially quantified coefficients. Similarly, the interval matrices  $B$  and  $C$  can be splits as  $B = B^\forall + B^\exists$ , and  $C = C^\forall + C^\exists$ , where  $B^\forall$ ,  $C^\forall$  are interval matrices comprising universally quantified coefficients, also  $B^\exists$ ,  $C^\exists$  are interval matrices concerns existentially quantified coefficients. We can decompose a matrix  $B = B^\forall + B^\exists$ . Then, a Matrix  $X$  is an AE solutions of

$$A \otimes X \otimes C \leq B \quad \text{if}$$

$$\forall A^\forall \in \mathbf{A}^\forall, \forall C^\forall \in \mathbf{C}^\forall, \forall B^\forall \in \mathbf{B}^\forall, \exists A^\exists \in \mathbf{A}^\exists, \exists C^\exists \in \mathbf{C}^\exists, \exists B^\exists \in \mathbf{B}^\exists$$

Such that

$$((A^\forall + A^\exists) \otimes (C^\forall + C^\exists)) \otimes X \leq B^\forall + B^\exists \quad (24)$$

In Max-plus algebra, by using for all-exist quantification of interval parameters, the matrices will be split as;

$$A = A^\forall \oplus A^\exists, B = B^\forall \oplus B^\exists, C = C^\forall \oplus C^\exists, \quad (25)$$

Where components in the matrices  $A^\forall, B^\forall, C^\forall$  on the positions associated with existential quantifiers are interval  $[\varepsilon, \varepsilon]$  and components in the matrices  $A^\exists, B^\exists, C^\exists$  on the positions associated with universal quantifiers are  $[\varepsilon, \varepsilon]$ .

Now the concept of AE solutions of interval linear system in classical algebra will extend to max-plus algebra.

**Definition 6:** A matrix  $X$  is an AE solution of equation (1) and (2). If

$$\forall A^\forall \in A^\forall, \forall C^\forall \in C^\forall, \forall B^\forall \in B^\forall, \forall D^\forall \in D^\forall, \forall F^\forall \in F^\forall, \forall E^\forall \in E^\forall, \exists A^\exists \in A^\exists, \exists C^\exists \in C^\exists, \exists B^\exists \in B^\exists, \exists D^\exists \in D^\exists, \exists F^\exists \in F^\exists, \exists E^\exists \in E^\exists$$

Such that

$$(A^\forall \oplus A^\exists) \otimes (C^\forall \oplus C^\exists) \otimes X \leq B^\forall \oplus B^\exists \quad (26)$$

$$(D^\forall \oplus D^\exists) \otimes (F^\forall \oplus F^\exists) \otimes X \leq B^\forall \oplus B^\exists \quad (27)$$

AE solutions covers many known solutions, especially, the most studied solution types (weak, strong, tolerance, control) as special cases [7-12].

In the same manner, the AE solutions for interval linear MPME will be define in similar way of Definition 7.

**Definition 7:**

A matrix  $X \in \mathbb{R}(n, s)$  is an AE solution of  $A \otimes X \otimes C = B$

if for  $\forall A^\forall \in A^\forall, \forall C^\forall \in C^\forall, \forall B^\forall \in B^\forall, A^\exists \in A^\exists, \exists C^\exists \in C^\exists, \exists B^\exists \in B^\exists$

such that

$$(A^\forall \oplus A^\exists) \otimes (C^\forall \oplus C^\exists) \otimes X = B^\forall \oplus B^\exists \quad (28)$$

For instance, consider the interval system over max-plus algebra

$$A = \begin{pmatrix} [2,6] & [1,3] \\ [1,4] & [3,4] \end{pmatrix} \otimes \begin{pmatrix} [2,5] & [0,4] \\ [3,7] & [7,9] \end{pmatrix} \otimes \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \\ = \begin{pmatrix} [3,15] & [6,17] \\ [2,13] & [5,15] \end{pmatrix} \quad (29)$$

$$\text{Let } A^\forall = \begin{pmatrix} [\varepsilon, \varepsilon] & [1,3] \\ [1,4] & [\varepsilon, \varepsilon] \end{pmatrix}, \quad C^\forall = \begin{pmatrix} [\varepsilon, \varepsilon] & [0,4] \\ [3,7] & [\varepsilon, \varepsilon] \end{pmatrix},$$

$$A^\exists = \begin{pmatrix} [2,6] & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & [3,4] \end{pmatrix},$$

$$C^\exists = \begin{pmatrix} [2,5] & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & [7,9] \end{pmatrix}, B^\forall = \begin{pmatrix} [3,15] & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & [5,15] \end{pmatrix},$$

$$B^\exists = \begin{pmatrix} [\varepsilon, \varepsilon] & [6,17] \\ [2,13] & [\varepsilon, \varepsilon] \end{pmatrix}$$

Then a matrix  $X$  is an AE solution to interval matrix equation (29) if for

$$\forall A^\forall = \begin{pmatrix} [\varepsilon, \varepsilon] & a_2 \\ a_3 & [\varepsilon, \varepsilon] \end{pmatrix} \in A^\forall,$$

$$\forall C^\forall = \begin{pmatrix} [\varepsilon, \varepsilon] & c_2 \\ c_3 & [\varepsilon, \varepsilon] \end{pmatrix} \in C^\forall,$$

$$\forall B^\forall = \begin{pmatrix} b_1 & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & b_4 \end{pmatrix} \in B^\forall \text{ and}$$

$$\exists A^\exists = \begin{pmatrix} a_2 & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & a_2 \end{pmatrix},$$

$$C^\exists = \begin{pmatrix} a_2 & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & a_2 \end{pmatrix},$$

$$B^\forall = \begin{pmatrix} a_2 & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & [a_2] \end{pmatrix},$$

$$B^\exists = \begin{pmatrix} [\varepsilon, \varepsilon] & b_2 \\ b_3 & [\varepsilon, \varepsilon] \end{pmatrix}.$$

$$((A^\forall \oplus A^\exists) \otimes (C^\forall \oplus C^\exists)) \otimes X = B^\forall \oplus B^\exists.$$

## 4.2 IME over max-plus algebra

Let recall the concept of AE solutions of interval inequalities in classical algebra, which characterized by specific order of logical quantifiers in the selecting predicate [15, 18, 19]. Each interval in interval linear system is associated with either the universal quantifier  $\forall$ , which is called A-uncertainty, or the existential quantifier  $\exists$ , which is called E-uncertainty. Therefore, let  $A \in \mathbb{R}(m, n)$ ,  $B \in \mathbb{R}(m, r)$  and  $C \in \mathbb{R}(s, r)$  be given matrices and we are looking for  $X \in \mathbb{R}(n, s)$ . Components in the matrix  $A^\forall$  on the positions associated with existential quantifier are intervals  $[0, 0]$ , and components in the matrix  $A^\exists$ , on the positions associated with universal quantifier are intervals  $[0, 0]$ . Thus, we can split the interval matrix as  $A = A^\forall + A^\exists$ , where  $A^\forall$  are the interval matrix comprising universally quantified coefficients, and  $A^\exists$  concerns existentially quantified coefficients. Similarly, the interval matrices  $B$  and  $C$  can be splits as  $B = B^\forall + B^\exists$ , and  $C = C^\forall + C^\exists$ , where  $B^\forall, C^\forall$  are interval matrices comprising of universally quantified coefficients. Also  $B^\exists, C^\exists$  are interval matrices concerns with existentially quantified coefficients. These can decompose a matrix  $B = B^\forall + B^\exists$ . Therefore, a Matrix  $X$  is an AE solutions of  $A \otimes X \otimes C \leq B$  if

$$\forall A^\forall \in A^\forall, \forall C^\forall \in C^\forall, \forall B^\forall \in B^\forall, \exists A^\exists \in A^\exists, \exists C^\exists \in C^\exists, \exists B^\exists \in B^\exists$$

Such that

$$((A^\forall + A^\exists) \otimes (C^\forall + C^\exists)) \otimes X \leq B^\forall + B^\exists$$

In Max-plus algebra, by using for all-exist quantification of interval parameters, the matrices will be split as  $A = A^\forall \oplus A^\exists, B = B^\forall \oplus B^\exists, C = C^\forall \oplus C^\exists$ , where components in the matrices  $A^\forall, B^\forall, C^\forall$  on the positions associated with existential quantifiers are interval  $[\varepsilon, \varepsilon]$  and components in the matrices  $A^\exists, B^\exists, C^\exists$  on the positions associated with universal quantifiers are  $[\varepsilon, \varepsilon]$ .

Now, the concept of AE solutions of interval linear system in classical algebra will be extending to max-plus algebra.



**Definition 8:** A matrix  $X$  is an AE solution of systems (1) and (2). If

$$\forall A^\vee \in \mathbf{A}^\vee, \forall C^\vee \in \mathbf{C}^\vee, \forall B^\vee \in \mathbf{B}^\vee, \forall D^\vee \in \mathbf{D}^\vee, \forall F^\vee \in \mathbf{F}^\vee, \forall E^\vee \in \mathbf{E}^\vee, \exists A^\exists \in \mathbf{A}^\exists, \exists C^\exists \in \mathbf{C}^\exists, \exists B^\exists \in \mathbf{B}^\exists, \exists D^\exists \in \mathbf{D}^\exists, \exists F^\exists \in \mathbf{F}^\exists, \exists E^\exists \in \mathbf{E}^\exists.$$

Such that

$$(A^\vee \oplus A^\exists) \otimes (C^\vee \oplus C^\exists) \otimes X \leq B^\vee \oplus B^\exists \quad (30)$$

$$(D^\vee \oplus D^\exists) \otimes (F^\vee \oplus F^\exists) \otimes X \leq B^\vee \oplus B^\exists \quad (31)$$

AE solutions covers many known solutions, especially, the most studied solution types (weak, strong, tolerance, control) as special cases [10, 11, 12], which was discuss in the next section.

In the same manner, the AE solutions for interval linear MPME are defined in similar way of Definition 9:

**Definition 9:** A matrix  $X \in \overline{\mathbb{R}}(n, s)$  is an AE solution of  $A \otimes X \otimes C = B$

$$\text{if for } \forall A^\vee \in \mathbf{A}^\vee, \forall C^\vee \in \mathbf{C}^\vee, \forall B^\vee \in \mathbf{B}^\vee, A^\exists \in \mathbf{A}^\exists, \exists C^\exists \in \mathbf{C}^\exists, \exists B^\exists \in \mathbf{B}^\exists$$

Such that

$$(A^\vee \oplus A^\exists) \otimes (C^\vee \oplus C^\exists) \otimes X = B^\vee \oplus B^\exists \quad (32)$$

For instance, consider the interval system over max-plus algebra

$$X = \begin{pmatrix} [2,6] & [1,3] \\ [1,4] & [3,4] \end{pmatrix} \otimes \begin{pmatrix} [2,5] & [0,4] \\ [3,7] & [7,9] \end{pmatrix} \otimes \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \\ = \begin{pmatrix} [3,15] & [6,17] \\ [2,13] & [5,15] \end{pmatrix} \quad (33)$$

Let

$$A^\vee = \begin{pmatrix} [\varepsilon, \varepsilon] & [1,3] \\ [1,4] & [\varepsilon, \varepsilon] \end{pmatrix}, \quad C^\vee = \begin{pmatrix} [\varepsilon, \varepsilon] & [0,4] \\ [3,7] & [\varepsilon, \varepsilon] \end{pmatrix}, \\ A^\exists = \begin{pmatrix} [2,6] & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & [3,4] \end{pmatrix}, \\ C^\exists = \begin{pmatrix} [2,5] & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & [7,9] \end{pmatrix} \\ \text{And } B^\vee = \begin{pmatrix} [3,15] & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & [5,15] \end{pmatrix}, B^\exists = \begin{pmatrix} [\varepsilon, \varepsilon] & [6,17] \\ [2,13] & [\varepsilon, \varepsilon] \end{pmatrix}$$

Then a matrix  $X$  is an AE solution to interval matrix equation (33) if for

$$\forall A^\vee = \begin{pmatrix} [\varepsilon, \varepsilon] & a_2 \\ a_3 & [\varepsilon, \varepsilon] \end{pmatrix} \in \mathbf{A}^\vee,$$

$$\forall C^\vee = \begin{pmatrix} [\varepsilon, \varepsilon] & c_2 \\ c_3 & [\varepsilon, \varepsilon] \end{pmatrix} \in \mathbf{C}^\vee,$$

$$\forall B^\vee = \begin{pmatrix} b_1 & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & b_4 \end{pmatrix} \in \mathbf{B}^\vee$$

$$\exists A^\exists = \begin{pmatrix} a_2 & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & a_2 \end{pmatrix}, C^\exists = \begin{pmatrix} a_2 & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & a_2 \end{pmatrix},$$

$$B^\vee = \begin{pmatrix} a_2 & [\varepsilon, \varepsilon] \\ [\varepsilon, \varepsilon] & a_2 \end{pmatrix}, B^\exists = \begin{pmatrix} [\varepsilon, \varepsilon] & b_2 \\ b_3 & [\varepsilon, \varepsilon] \end{pmatrix}.$$

$$((A^\vee \oplus A^\exists) \otimes (C^\vee \oplus C^\exists)) \otimes X = B^\vee \oplus B^\exists.$$

### 4.3 Characterization of AE solutions for interval system of MPME

Consider an interval matrix equation of the following form

$$A \otimes X \otimes C = B \quad (34)$$

Where  $A \in \overline{\mathbb{R}}(m_1, n)$ ,  $B \in \overline{\mathbb{R}}(m_1, r)$ ,  $C \in \overline{\mathbb{R}}(s_1, r)$ .

Full characterization of AE solutions to interval equation (20) will be investigated here. Regards to that, also a constructive result described in Theorem 1 concerning about the dependency property which is worth stating independently was established.

**Theorem 1:**

A matrix  $X \in \overline{\mathbb{R}}(n, r)$  is an AE solution of interval linear MPME  $A \otimes X \otimes C = B$  if and only if:

$$(\overline{A^\vee} \oplus \underline{A^\exists}) \otimes X \otimes (\overline{C^\vee} \oplus \underline{C^\exists}) \geq (\underline{B^\vee} \oplus \overline{B^\exists}) \quad (35)$$

$$(\underline{A^\vee} \oplus \overline{A^\exists}) \otimes X \otimes (\underline{C^\vee} \oplus \overline{C^\exists}) \leq (\overline{B^\vee} \oplus \underline{B^\exists}) \quad (36)$$

**Proof.** Assume that  $X \in \overline{\mathbb{R}}(n, r)$  is an AE solutions of  $A \otimes X \otimes C = B$ . Then for

$$\forall A^\vee \in \mathbf{A}^\vee, \forall B^\vee \in \mathbf{B}^\vee, \text{ and } \forall C^\vee \in \mathbf{C}^\vee \text{ the equation}$$

$$(A^\vee \oplus A^\exists) \otimes X \otimes (C^\vee \oplus C^\exists) = (B^\vee \oplus B^\exists) \quad (37)$$

Is solvable for some  $B^\exists = B_0^\exists, A^\exists = A_0^\exists, C^\exists = C_0^\exists$ .

Due to the isotone properties and distributive property presented in Lemma 3 and Lemma 4 for  $\forall A^\vee \in \mathbf{A}^\vee, \forall B^\vee \in \mathbf{B}^\vee, \text{ and } \forall C^\vee \in \mathbf{C}^\vee$ , have

$$(A^\vee \oplus \underline{A^\exists}) \otimes X \otimes (C^\vee \oplus \underline{C^\exists}) \leq (A^\vee \oplus A^\exists) \otimes X \otimes (C^\vee \oplus C^\exists)$$

$$= B^\vee \oplus B^\exists \leq B^\vee \oplus \overline{B^\exists} \quad (38)$$

Particularly, let  $A^\vee = \overline{A^\vee}, B^\vee = \overline{B^\vee}, C^\vee = \overline{C^\vee}$  from equation (38), then:

$$(\overline{A^\vee} \oplus \underline{A^\exists}) \otimes X \otimes (\overline{C^\vee} \oplus \underline{C^\exists}) \leq \underline{B^\vee} \oplus \overline{B^\exists} \quad (39)$$

On the other hand, according to the isotone properties and distributive property presented in Lemma 3 and Lemma 4, for  $\forall A^\vee \in \mathbf{A}^\vee, \forall B^\vee \in \mathbf{B}^\vee, \text{ and } \forall C^\vee \in \mathbf{C}^\vee$ , we also have

$$(\underline{A^\vee} \oplus \overline{A^\exists}) \otimes X \otimes (\underline{C^\vee} \oplus \overline{C^\exists}) \geq (A^\vee \oplus A^\exists) \otimes X \otimes (C^\vee \oplus C^\exists) \quad (40)$$

$$= B^\vee \oplus B^\exists \geq B^\vee \oplus \underline{B^\exists} \quad (41)$$

Particularly, let  $A^\vee = \overline{A^\vee}, B^\vee = \overline{B^\vee}$  in (41), have

$$(\underline{A^\vee} \oplus \overline{A^\exists}) \otimes X \otimes (\underline{C^\vee} \oplus \overline{C^\exists}) \geq \overline{B^\vee} \oplus \underline{B^\exists} \quad (42)$$

### 4.4 Special cases of AE solutions for interval system of MPME

The traditional definition of a solution to interval linear system contains; Weak solutions, strong solutions, tolerance solutions and control solutions. Similarly, define

an equivalent concept of solutions for interval max-plus matrix linear system.

**Definition 10:** An unknown matrix  $X \in \overline{\mathbb{R}}(n, r)$  is called

a. A weak solution of equation (20), (21) if inequalities (22), (23) hold for some

$A \in \mathbf{A}, B \in \mathbf{B}, C \in \mathbf{C}$  and  $D \in \mathbf{D}, E \in \mathbf{E}, F \in \mathbf{F}$ ;

b. A strong solution of equation (20), (21) if inequalities (22), (23) hold for some

$A \in \mathbf{A}, B \in \mathbf{B}, C \in \mathbf{C}$  and  $D \in \mathbf{D}, E \in \mathbf{E}, F \in \mathbf{F}$ ;

c. A tolerance solution of equation (20), (21) if for each  $A \in \mathbf{A}, C \in \mathbf{C}$ ,

$D \in \mathbf{D}, F \in \mathbf{F}$ , Inequalities (22), (23) hold for some  $B \in \mathbf{B}, E \in \mathbf{E}$ ;

d. A control solution of equation (20), (21) if for each  $B \in \mathbf{B}, E \in \mathbf{E}$ , inequalities (22), (23) hold for some

$A \in \mathbf{A}, C \in \mathbf{C}$  and  $D \in \mathbf{D}, E \in \mathbf{E}, F \in \mathbf{F}$ ;

a. A weak solution of equation (22) and (23) hold for some

$A \in \mathbf{A}, B \in \mathbf{B}, C \in \mathbf{C}$  and  $D \in \mathbf{D}, E \in \mathbf{E}, F \in \mathbf{F}$ ;

b. A strong solution of equation (22) and (23) hold for all

$A \in \mathbf{A}, B \in \mathbf{B}, C \in \mathbf{C}$  and  $D \in \mathbf{D}, E \in \mathbf{E}, F \in \mathbf{F}$ ;

c. A tolerance solution of equation (22) if for each  $A \in \mathbf{A}, C \in \mathbf{C}$  (23) hold for some  $B \in \mathbf{B}$ ;

d. A control solution of equation (22) if for each  $B \in \mathbf{B}$ , (23) hold for some  $A \in \mathbf{A}, C \in \mathbf{C}$ .

From the definition of AE solutions, it is easy to obtain that the weak solution, strong solution, tolerance solution and control solution are special cases of AE solutions. Necessary and sufficient conditions for checking weak and strong solvability of interval systems in max-plus algebra were formulated in [8, 9], it is found that, the type of solvability of interval linear max-plus systems are also special cases in Theorem 1.

The characterizations of the weak and strong solutions of equations (20) and (21) are proposed in the Corollary 1 and Corollary 2 below.

**Corollary 1:** An unknown matrix  $X \in \overline{\mathbb{R}}(n, r)$  is weak solution of the equations 1 and 2 if and only if

$$\underline{A} \otimes \underline{C} \otimes X \leq \underline{B}$$

$$\overline{D} \otimes \overline{F} \otimes X \geq \overline{E}$$

**Proof:** The assertion follows immediately from Theorem 4.1 if we set  $\mathbf{A}^\vee = \varepsilon$ ,

$$\mathbf{C}^\vee = \varepsilon, \mathbf{B}^\vee = \varepsilon \text{ and } \mathbf{D}^\vee = \varepsilon, \mathbf{F}^\vee = \varepsilon, \mathbf{E}^\vee = \varepsilon$$

**Corollary 2:** An unknown matrix  $X \in \overline{\mathbb{R}}(n, r)$  is strong solution of the equation 1 and 2 if and only if

$$\overline{A} \otimes \overline{C} \otimes X \leq \overline{B}$$

$$\underline{D} \otimes \underline{F} \otimes X \geq \underline{E}.$$

**Proof:** The assertion follows immediately from theorem.1 if

$$\mathbf{A}^\exists = \varepsilon, \mathbf{C}^\exists = \varepsilon, \mathbf{B}^\exists = \varepsilon \text{ and } \mathbf{D}^\exists = \varepsilon, \mathbf{F}^\exists = \varepsilon, \mathbf{E}^\exists = \varepsilon$$

The characterizations of the weak and strong solutions of equation (20) are proposed in Corollary 3 and Corollary 4 below

**Corollary 3:** an unknown matrix  $X \in \overline{\mathbb{R}}(n, r)$  is a weak solution of the equation (20), if and only if:

$$\underline{A} \otimes \underline{C} \otimes X \leq \underline{B}$$

$$\overline{A} \otimes \overline{C} \otimes X \geq \overline{B}.$$

**Proof:** The assertion follows immediately from Theorem 1 if we set  $\mathbf{A}^\vee = \varepsilon, \mathbf{C}^\vee = \varepsilon, \mathbf{B}^\vee = \varepsilon$ .

**Example 1:** Given

$$A = \begin{pmatrix} [2,4] & [3,5] \\ [4,7] & [6,9] \end{pmatrix}, C = \begin{pmatrix} [7,9] & [3,6] \\ [4,6] & [8,11] \end{pmatrix},$$

$$B = \begin{pmatrix} [8,15] & [3,18] \\ [4,17] & [5,20] \end{pmatrix}$$

By Corollary 3, it can be check that matrix  $X = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$  is a weak solution to equation (19)

**Solution:**

Let

$$\overline{A} = \begin{pmatrix} 4 & 5 \\ 7 & 9 \end{pmatrix}, \underline{A} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}, \overline{C} = \begin{pmatrix} 9 & 6 \\ 6 & 11 \end{pmatrix},$$

$$\underline{C} = \begin{pmatrix} 7 & 3 \\ 4 & 8 \end{pmatrix} \text{ and}$$

$$\overline{B} = \begin{pmatrix} 15 & 18 \\ 17 & 20 \end{pmatrix}, \underline{B} = \begin{pmatrix} 8 & 3 \\ 4 & 5 \end{pmatrix}, \text{ then}$$

$$\underline{A} \otimes \underline{C} \otimes X = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \otimes \begin{pmatrix} 7 & 3 \\ 4 & 8 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \otimes 7 \oplus 3 \otimes 4 & 2 \otimes 3 \oplus 3 \otimes 8 \\ 4 \otimes 7 \oplus 6 \otimes 4 & 4 \otimes 3 \oplus 6 \otimes 8 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 9 \oplus 7 & 5 \oplus 11 \\ 11 \oplus 10 & 7 \oplus 14 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 11 \\ 11 & 14 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 9 \otimes 1 \oplus 11 \otimes 2 & 9 \otimes 3 \oplus 11 \otimes 4 \\ 11 \otimes 1 \oplus 14 \otimes 2 & 11 \otimes 3 \oplus 14 \otimes 4 \end{pmatrix}$$

$$= \begin{pmatrix} 10 \oplus 13 & 12 \oplus 15 \\ 12 \oplus 16 & 14 \oplus 18 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 15 \\ 16 & 18 \end{pmatrix} \leq \begin{pmatrix} 15 & 18 \\ 17 & 20 \end{pmatrix} = \overline{B}$$

Thus,  $\underline{A} \otimes \underline{C} \otimes X \leq \underline{B}$

$$\overline{A} \otimes \overline{C} \otimes X = \begin{pmatrix} 4 & 5 \\ 7 & 9 \end{pmatrix} \otimes \begin{pmatrix} 9 & 6 \\ 6 & 11 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \otimes 9 \oplus 5 \otimes 6 & 4 \otimes 6 \oplus 5 \otimes 11 \\ 7 \otimes 9 \oplus 9 \otimes 6 & 7 \otimes 6 \oplus 9 \otimes 11 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 13 \oplus 11 & 10 \oplus 16 \\ 16 \oplus 15 & 13 \oplus 20 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 13 & 16 \\ 16 & 20 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 13 \otimes 1 \oplus 16 \otimes 2 & 13 \otimes 3 \oplus 16 \otimes 4 \\ 16 \otimes 1 \oplus 20 \otimes 2 & 15 \otimes 3 \oplus 20 \otimes 4 \end{pmatrix}$$

$$= \begin{pmatrix} 14 \oplus 18 & 16 \oplus 20 \\ 17 \oplus 22 & 18 \oplus 24 \end{pmatrix} = \begin{pmatrix} 18 & 20 \\ 22 & 24 \end{pmatrix} \geq \begin{pmatrix} 8 & 3 \\ 4 & 5 \end{pmatrix} = \underline{B}.$$

Thus,  $\overline{A} \otimes \overline{C} \otimes X \geq \underline{B}$ .

**Corollary 4:** An unknown matrix  $X \in \overline{\mathbb{R}}(n, r)$  is strong solution of the equation 20 if and only if

$$\overline{A} \otimes \overline{C} \otimes X \leq \underline{B} \text{ and}$$

$$\underline{A} \otimes \underline{C} \otimes X \geq \overline{B}.$$

**Proof.** The assertion follows immediately from Theorem 1 if

$$A^{\exists} = \varepsilon$$

$$C^{\exists} = \varepsilon, B^{\exists} = \varepsilon.$$

**Example 4:** Given

$$A = \begin{pmatrix} [2,4] & [3,5] \\ [4,7] & [6,9] \end{pmatrix}, C = \begin{pmatrix} [7,9] & [3,6] \\ [4,6] & [8,11] \end{pmatrix},$$

$$B = \begin{pmatrix} [14,17] & [15,17] \\ [16,18] & [20,21] \end{pmatrix}$$

By Corollary 4, it can be checked that matrix  $X = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$  is a strong solution to equation (20).

Solution

Since,

$$\overline{A} = \begin{pmatrix} 4 & 5 \\ 7 & 9 \end{pmatrix}, \underline{A} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}, \overline{C} = \begin{pmatrix} 9 & 6 \\ 6 & 11 \end{pmatrix},$$

$$\underline{C} = \begin{pmatrix} 7 & 3 \\ 4 & 8 \end{pmatrix} \text{ and}$$

$$\overline{B} = \begin{pmatrix} 17 & 17 \\ 18 & 21 \end{pmatrix}, \underline{B} = \begin{pmatrix} 14 & 15 \\ 16 & 20 \end{pmatrix}, \text{ then}$$

$$\begin{aligned} \underline{A} \otimes \underline{C} \otimes X &= \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \otimes \begin{pmatrix} 7 & 3 \\ 4 & 8 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2 \otimes 7 \oplus 3 \otimes 4 & 2 \otimes 3 \oplus 3 \otimes 8 \\ 4 \otimes 7 \oplus 6 \otimes 4 & 4 \otimes 3 \oplus 6 \otimes 8 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 9 \oplus 7 & 5 \oplus 11 \\ 11 \oplus 10 & 7 \oplus 14 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 11 \\ 11 & 14 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 9 \otimes 1 \oplus 11 \otimes 2 & 9 \otimes 3 \oplus 11 \otimes 4 \\ 11 \otimes 1 \oplus 14 \otimes 2 & 11 \otimes 3 \oplus 14 \otimes 4 \end{pmatrix}$$

$$= \begin{pmatrix} 10 \oplus 13 & 12 \oplus 15 \\ 12 \oplus 16 & 14 \oplus 18 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 15 \\ 16 & 18 \end{pmatrix} \leq \begin{pmatrix} 14 & 15 \\ 16 & 20 \end{pmatrix} = \underline{B}$$

Thus,  $\underline{A} \otimes \underline{C} \otimes X \leq \underline{B}$

$$\overline{A} \otimes \overline{C} \otimes X = \begin{pmatrix} 4 & 5 \\ 7 & 9 \end{pmatrix} \otimes \begin{pmatrix} 9 & 6 \\ 6 & 11 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \otimes 9 \oplus 5 \otimes 6 & 4 \otimes 6 \oplus 5 \otimes 11 \\ 7 \otimes 9 \oplus 9 \otimes 6 & 7 \otimes 6 \oplus 9 \otimes 11 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 13 \oplus 11 & 10 \oplus 16 \\ 16 \oplus 15 & 13 \oplus 20 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 16 \\ 16 & 20 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 13 \otimes 1 \oplus 16 \otimes 2 & 13 \otimes 3 \oplus 16 \otimes 4 \\ 16 \otimes 1 \oplus 20 \otimes 2 & 16 \otimes 3 \oplus 20 \otimes 4 \end{pmatrix}$$

$$= \begin{pmatrix} 14 \oplus 18 & 16 \oplus 19 \\ 17 \oplus 22 & 19 \oplus 24 \end{pmatrix}$$

$$= \begin{pmatrix} 18 & 19 \\ 22 & 24 \end{pmatrix} \geq \begin{pmatrix} 17 & 17 \\ 18 & 21 \end{pmatrix} = \overline{B}.$$

Thus  $\overline{A} \otimes \overline{C} \otimes X \geq \overline{B}$

Besides weak solvability and strong solvability, the following new characterizations of tolerance and control solutions are the simple consequences of Theorem 1.

The Characterizations of the tolerance and control solutions of system (13), (14) are proposed in Corollary 5 and Corollary 6 below

**Corollary 5:** An unknown matrix  $X \in \overline{\mathbb{R}}(n, r)$  is a tolerable solution of the equation 13 and 14 if and only if

$$\overline{A} \otimes X \otimes \overline{C} \leq \overline{B},$$

$$\underline{D} \otimes X \otimes \underline{E} \geq \underline{F}.$$

**Proof:** The assertion follows immediately from Theorem 1 if set:

$$A^{\exists} = \varepsilon, C^{\exists} = \varepsilon, B^{\forall} = \varepsilon \text{ and } D^{\exists} = \varepsilon, F^{\exists} = \varepsilon, E^{\forall} = \varepsilon.$$

**Corollary 6:** An unknown matrix  $X \in \overline{\mathbb{R}}(n, r)$  is a control solution of the equation 13 and 14 if and only if

$$\underline{A} \otimes X \otimes \underline{C} \leq \underline{B}$$

$$\overline{D} \otimes X \otimes \overline{E} \geq \overline{F}.$$

**Proof:** The assertion follows immediately from Theorem 1 if

$$A^{\forall} = \varepsilon$$

$$C^{\forall} = \varepsilon, B^{\exists} = \varepsilon \text{ and } D^{\forall} = \varepsilon, F^{\forall} = \varepsilon, E^{\exists} = \varepsilon.$$

The characterizations of the tolerance and control solutions of equation (20) are proposed in Corollary 7 and Corollary 8 below

**Corollary 7:** An unknown matrix  $X \in \overline{\mathbb{R}}(n, r)$  is a tolerable solution of the equation 2 if and only if

$$\overline{A} \otimes X \otimes \overline{C} \leq \overline{B},$$

$$\underline{A} \otimes X \otimes \underline{C} \geq \underline{B}.$$

**Proof:** The assertion follows immediately from Theorem 1 if we set:

$$A^{\exists} = \varepsilon$$

$$C^{\exists} = \varepsilon, B^{\forall} = \varepsilon.$$

**Example 5:** Given

$$A = \begin{pmatrix} [5,8] & [3,7] \\ [4,6] & [4,5] \end{pmatrix}, C = \begin{pmatrix} [7,9] & [5,8] \\ [4,7] & [3,6] \end{pmatrix},$$

$$B = \begin{pmatrix} [14,22] & [8,20] \\ [12,19] & [7,18] \end{pmatrix}$$

By Corollary 7, it can be seen that matrix  $X = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$  is a tolerance solution to system (20).



Solution: Since,

$$\begin{aligned}\bar{A} &= \begin{pmatrix} 8 & 7 \\ 6 & 5 \end{pmatrix}, \underline{A} = \begin{pmatrix} 5 & 3 \\ 4 & 4 \end{pmatrix}, \bar{C} = \begin{pmatrix} 9 & 8 \\ 7 & 6 \end{pmatrix}, \\ \underline{C} &= \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix} \text{ and } \bar{B} = \begin{pmatrix} 22 & 20 \\ 19 & 18 \end{pmatrix}, \underline{B} = \begin{pmatrix} 14 & 8 \\ 12 & 7 \end{pmatrix}, \text{ then} \\ \bar{A} \otimes \bar{C} \otimes X &= \begin{pmatrix} 8 & 7 \\ 6 & 5 \end{pmatrix} \otimes \begin{pmatrix} 9 & 8 \\ 7 & 6 \end{pmatrix} \otimes \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 8 \otimes 9 \oplus 7 \otimes 7 & 8 \otimes 8 \oplus 7 \otimes 6 \\ 6 \otimes 9 \oplus 5 \otimes 7 & 6 \otimes 8 \oplus 5 \otimes 6 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 17 \oplus 14 & 16 \oplus 13 \\ 15 \oplus 12 & 14 \oplus 11 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 17 & 16 \\ 15 & 14 \end{pmatrix} \otimes \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 17 \otimes 1 \oplus 16 \otimes 4 & 17 \otimes 2 \oplus 16 \otimes 3 \\ 15 \otimes 1 \oplus 14 \otimes 4 & 15 \otimes 2 \oplus 14 \otimes 3 \end{pmatrix} \\ &= \begin{pmatrix} 18 \oplus 20 & 19 \oplus 19 \\ 16 \oplus 18 & 17 \oplus 17 \end{pmatrix} = \begin{pmatrix} 20 & 19 \\ 18 & 17 \end{pmatrix} \leq \begin{pmatrix} 22 & 20 \\ 19 & 18 \end{pmatrix}\end{aligned}$$

Thus,  $\bar{A} \otimes X \otimes \bar{C} \leq \bar{B}$ .

Similarly,

$$\begin{aligned}\underline{A} \otimes \underline{C} \otimes X &= \begin{pmatrix} 5 & 3 \\ 4 & 4 \end{pmatrix} \otimes \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix} \otimes \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 5 \otimes 7 \oplus 3 \otimes 4 & 5 \otimes 5 \oplus 3 \otimes 3 \\ 4 \otimes 7 \oplus 4 \otimes 4 & 4 \otimes 5 \oplus 4 \otimes 3 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 12 \oplus 7 & 10 \oplus 6 \\ 11 \oplus 8 & 9 \oplus 7 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 12 & 10 \\ 11 & 9 \end{pmatrix} \otimes \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 12 \otimes 1 \oplus 10 \otimes 4 & 12 \otimes 2 \oplus 10 \otimes 3 \\ 11 \otimes 1 \oplus 9 \otimes 4 & 11 \otimes 2 \oplus 9 \otimes 3 \end{pmatrix} \\ &= \begin{pmatrix} 13 \oplus 14 & 14 \oplus 13 \\ 12 \oplus 13 & 13 \oplus 12 \end{pmatrix} \\ &= \begin{pmatrix} 14 & 14 \\ 13 & 13 \end{pmatrix} \geq \begin{pmatrix} 14 & 8 \\ 12 & 7 \end{pmatrix},\end{aligned}$$

Thus,  $\underline{A} \otimes X \otimes \underline{C} \geq \underline{B}$ .

**Corollary 8:** An unknown matrix  $X \in \mathbb{R}(n, r)$  is a control solution of the equation 20, if and only if

$$\underline{A} \otimes X \otimes \underline{C} \geq \underline{B}$$

$$\bar{A} \otimes X \otimes \bar{C} \leq \bar{B}.$$

**Proof:** The assertion follows immediately from Theorem 1: if  $A^\vee = \varepsilon$ ,  $C^\vee = \varepsilon$ ,  $B^\exists = \varepsilon$ .

**Example 6:** Given

$$\begin{aligned}A &= \begin{pmatrix} [3,5] & [4,12] \\ [2,9] & [5,6] \end{pmatrix}, C = \begin{pmatrix} [4,6] & [3,10] \\ [2,7] & [6,7] \end{pmatrix}, \\ B &= \begin{pmatrix} [12,25] & [10,23] \\ [13,24] & [9,30] \end{pmatrix}\end{aligned}$$

By Corollary 8, we can show that matrix  $X = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$  is a control solution to equation (20).

Solution: Since,

$$\begin{aligned}\bar{A} &= \begin{pmatrix} 5 & 12 \\ 9 & 6 \end{pmatrix}, \underline{A} = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}, \bar{C} = \begin{pmatrix} 6 & 10 \\ 7 & 7 \end{pmatrix}, \\ \underline{C} &= \begin{pmatrix} 4 & 3 \\ 2 & 6 \end{pmatrix} \text{ and } \bar{B} = \begin{pmatrix} 25 & 23 \\ 24 & 30 \end{pmatrix}, \underline{B} = \begin{pmatrix} 12 & 10 \\ 13 & 9 \end{pmatrix}, \\ \text{then}\end{aligned}$$

$$\begin{aligned}A \otimes C \otimes X &= \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} \otimes \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} \otimes \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 7 \oplus 6 & 6 \oplus 10 \\ 6 \oplus 7 & 5 \oplus 11 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 7 & 11 \end{pmatrix} \otimes \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 7 \otimes 1 \oplus 10 \otimes 4 & 7 \otimes 2 \oplus 10 \otimes 3 \\ 7 \otimes 1 \oplus 11 \otimes 4 & 7 \otimes 2 \oplus 11 \otimes 3 \end{pmatrix} \\ &= \begin{pmatrix} 8 \oplus 14 & 9 \oplus 13 \\ 8 \oplus 15 & 9 \oplus 14 \end{pmatrix} = \begin{pmatrix} 14 & 13 \\ 15 & 14 \end{pmatrix} \geq \begin{pmatrix} 12 & 10 \\ 13 & 9 \end{pmatrix}\end{aligned}$$

Thus,  $\underline{A} \otimes X \otimes \underline{C} \geq \underline{B}$ .

$$\begin{aligned}\bar{A} \otimes \bar{C} \otimes X &= \begin{pmatrix} 5 & 12 \\ 9 & 6 \end{pmatrix} \otimes \begin{pmatrix} 6 & 10 \\ 7 & 7 \end{pmatrix} \otimes \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 5 \otimes 6 \oplus 12 \otimes 7 & 5 \otimes 10 \oplus 12 \otimes 7 \\ 9 \otimes 6 \oplus 6 \otimes 7 & 9 \otimes 10 \oplus 6 \otimes 7 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 11 \oplus 19 & 15 \oplus 19 \\ 15 \oplus 13 & 19 \oplus 13 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 19 & 19 \\ 15 & 19 \end{pmatrix} \otimes \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 19 \otimes 1 \oplus 19 \otimes 4 & 19 \otimes 2 \oplus 19 \otimes 3 \\ 15 \otimes 1 \oplus 19 \otimes 4 & 15 \otimes 2 \oplus 19 \otimes 3 \end{pmatrix} \\ &= \begin{pmatrix} 20 \oplus 23 & 21 \oplus 22 \\ 16 \oplus 23 & 17 \oplus 22 \end{pmatrix} = \begin{pmatrix} 23 & 22 \\ 23 & 22 \end{pmatrix} \leq \begin{pmatrix} 25 & 23 \\ 24 & 30 \end{pmatrix} = \bar{B}.\end{aligned}$$

Thus,  $\bar{A} \otimes X \otimes \bar{C} \leq \bar{B}$ .

## 5. Conclusion

In this paper, a new concept of AE solution to interval max-plus linear system using matrix equations has been introduced. Consequently, comprehensive characterizations of AE solutions for interval max-plus matrix equation (MPME) were developed, covering both linear inequalities and linear equations. Additionally, special cases including weak, strong, tolerance and control solutions were discussed.

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